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STRAIN LOCALIZATION AND BIFURCATION IN A NONLOCAL CONTINUUM

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Abstract—The conditions for localization and wave propagation in a strain softening material described by a nonlocal damage-based constitutive relation are derived in closed form. Localization is understood as a bifurcation into a harmonic mode. The criterion for bifurcation is reduced to the classical form of singularity of a pseudo "acoustic tensor"; this tensor is not a material property as it involves the wavelength of the bifurcation mode through the Fourier transform of the weight function used in the definition of the nonlocal damage. A geometrical solution is provided to analyse localization. The conditions for the onset of bifurcation are found to coincide in the nonlocal and in the corresponding local cases. In the nonlocal continuum, the wavelength of the localization mode is constrained to remain below a threshold which is proportional to the characteristic length of the continuum. The analysis in dynamics exhibits the well-known property of wave dispersion. In some instances, i.e. for large wavelength modes, wave celerities become imaginary, but waves with a sufficiently short wavelength are found to propagate during softening in all the situations.

1. INTRODUCTION

Failure analyses of structures made of progressively fracturing rate-independent materials lead to fundamental difficulties due to the possible nonassociativity of the evolution equations but also to the loss of positive definiteness of the tangent material stiffness operator. This last type of behaviour, which is the focus of the present analyses carried out under the small strain assumption, is called strain-softening. It favours localization instabilities (Rudnicki and Rice, 1975; Rice, 1976) and more importantly, ill-posedness of boundary value problems.

In statics the partial differential equations governing equilibrium do not remain elliptic. This result was derived for the linearized rate equilibrium problem, considering what is commonly denoted as the linear comparison solid (Hill, 1959). Loss of ellipticity corresponds to a situation in which either the number of linearly independent solutions to the equilibrium equations is infinite and (or) these solutions do not depend continuously on the data as formulated in a general context by Benallal *et al.* (1988, 1991). Among the possible solutions is the case where the rate of deformation is discontinuous in the solid and where the total energy consumption remains constant. Path stability considerations indicate that this particular strain localization solution is expected and consequently failure occurs without energy dissipation (Bazant, 1988).

In dynamics, the differential equations of motion are hyperbolic when the material does not soften, and they may become parabolic or elliptic in the presence of strain-softening. Again, the initial value problem becomes ill-posed as the wave velocity becomes imaginary (Hadamard, 1903). The analytical solution derived by Bazant and Belytschko (1985) for the interference of constant strain waves in a strain softening rod exhibits the same feature as in statics, i.e. failure without energy dissipation.

The difficulties associated to strain-softening have led a number of researchers to investigate nonconventional constitutive relations, the so-called "localization limiters". In most cases, such constitutive relations incorporate an enrichment of the continuum which introduces an internal (or characteristic) length, and thus, enforce the energy dissipation to remain a finite but non zero quantity as failure occurs (Simo, 1989). In the micropolar continuum (Muhlhaus and Vardoulakis, 1987; De Borst, 1991), the additional degree of freedom (micro-rotation) brings in an internal length scale which prevents the strain rate to become discontinuous under certain loading types (Steinmann and Willam, 1991). The

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same feature is attached to nonlocal constitutive relations written in a gradient format such as the gradient dependent plasticity models by Lasry and Belytschko (1988), Schreyer (1990), and De Borst and Muhlhaus (1992). Another possibility explored by Needleman (1988), Loret and Prevost (1990), and Sluys and De Borst (1992) is the implementation of time dependent strain-softening relation. In this case, which deals only with dynamics, the equations of motion remain hyperbolic during softening although in some instances the term which regularizes the equation can vanish for sufficiently high values of the viscoplastic strain (Sluys, 1992).

In order to obtain a complete picture of the properties attached to localization limiters, strain localization in a medium following a nonlocal integral constitutive relation needs to be investigated. Pijaudier-Cabot and Bodé (1992) carried out such an analysis although the results presented hold for a two-dimensional medium subjected to radial loadings only. It was also demonstrated that the strain rate field cannot become discontinuous as bifurcation occurs (which is obvious for gradient dependent models). Leblond *et al.* (1992) obtained such a result considering a nonlocal Gurson model. In this paper, we extend the analysis by Pijaudier-Cabot and Bodé (1992) and consider the general, three-dimensional case of a continuum following a nonlocal damage model (Pijaudier-Cabot and Bazant, 1987) in statics and dynamics. The paper is organized as follows: In Section 2, the damage model is briefly recalled. The rate constitutive relations for the linear comparison solid are derived and the conditions for bifurcation from a homogeneous state are derived. Section 3 deals with strain localization in statics. The criterion for bifurcation is derived under the restriction that the solid considered is infinite or sufficiently large so that boundary layer effects can be neglected. The results are compared with the localization analysis in a usual (local) continuum and the effect of the characteristic length on localization pointed out. In Section 4, the same analysis is performed in dynamics.

2. RATE EQUILIBRIUM EQUATIONS

2.1. *Constitutive relations*

We will use in the following the scalar continuous damage model proposed by Pijaudier-Cabot and Bazant (1987). Despite its simplicity, this model bears the essential characteristics pertaining to a nonlocal integral model. Similar developments could be performed with a nonlocal plasticity model or with a nonlocal Gurson model (Leblond *et al.*, 1992). The constitutive relations read:

$$\boldsymbol{\sigma} = (1-D)\mathbb{E}:\boldsymbol{\epsilon}, \tag{1}$$

where the colon denotes the contracted tensorial product, $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{\epsilon}$ is the strain tensor, $\mathbb{E}$ is the Hooke tensor of the undamaged material, and D is the damage variable. The growth of damage is defined classically by a loading function f:

$$f(\bar{y}, D) = \int_0^{\bar{y}} F(z)\,\mathrm{d}z - D, \tag{2}$$

where F is a function which is deduced from experimental data and $\bar{y}(\mathbf{x})$ is the average energy release rate due to damage at point $\mathbf{x}$ of the solid:

$$\bar{y}(\mathbf{x}) = \int_V \Psi(\mathbf{x}-\mathbf{s})\,y(\mathbf{s})\,\mathrm{d}\mathbf{s}. \tag{3}$$

$\Psi(\mathbf{x}-\mathbf{s})$ is a normalized weighting function, V is the volume of the solid and $y(\mathbf{s})$ is the energy release rate due to damage at point $\mathbf{s}$ defined by:

$$y(\mathbf{s}) = \tfrac{1}{2}\boldsymbol{\epsilon}(\mathbf{s}):\mathbb{E}:\boldsymbol{\epsilon}(\mathbf{s}). \tag{4}$$

The weighting function Ψ and the function F need not be specified at this stage but they will be detailed in the numerical examples. The evolution law is usually prescribed in nonassociated plasticity or damage:

$$\dot{D} = \delta \frac{\partial g}{\partial \bar{y}} \tag{5}$$

with the classical Kuhn–Tucker conditions $\delta \geqslant 0$, $f \leqslant 0$ and $\delta \dot{f} = 0$. $\dot{X}$ indicates the time derivative of X, g is the evolution potential controlling the growth of damage and δ is the damage multiplier. In this paper, we have chosen for the sake of simplicity $g = \bar{y}$.

Consider now an initial state of equilibrium denoted by the state variables D_0 and $\boldsymbol{\epsilon}_0$. The rate constitutive relations describing the behaviour of the material from this state are obtained by differentiating (1) with respect to time:

$$\dot{\boldsymbol{\sigma}} = (1 - D_0)\mathbb{E} : \dot{\boldsymbol{\epsilon}} - \dot{D}\mathbb{E} : \boldsymbol{\epsilon}_0, \tag{6}$$

Using the damage law (5), eqn (6) becomes:

$$\dot{\boldsymbol{\sigma}}(\mathbf{x}) = (1 - D_0)\mathbb{E} : \dot{\boldsymbol{\epsilon}}(\mathbf{x}) - F(\bar{y}_0)\mathbb{E} : \boldsymbol{\epsilon}_0 \int_V \Psi(\mathbf{s})\boldsymbol{\epsilon}_0(\mathbf{s}+\mathbf{x}) : \mathbb{E} : \dot{\boldsymbol{\epsilon}}(\mathbf{s}+\mathbf{x})\,\mathrm{d}\mathbf{s} \tag{7}$$

at the points of the solid where damage grows and

$$\dot{\boldsymbol{\sigma}}(\mathbf{x}) = (1 - D_0)\mathbb{E} : \dot{\boldsymbol{\epsilon}}(\mathbf{x}) \tag{8}$$

elsewhere.

2.2. *Equations of motion*

Since the constitutive relations describing the behaviour of the material are nonlinear, the equations of motion ought to be a set of nonlinear integro-differential equations. Upon linearization of the equation of motion about the initial state $(\boldsymbol{\epsilon}_0, D_0)$, the momentum equations become:

$$\operatorname{div} \dot{\boldsymbol{\sigma}}(\mathbf{x}) = \rho \frac{\partial^2 \mathbf{v}}{\partial t^2}, \tag{9}$$

in which $\mathbf{v}$ is the time derivative of the perturbation applied to the initial state. Equations (9) are still nonlinear according to the rate constitutive relations [eqns (7), (8)]. The linearization is performed under the assumption that $\dot{f} = 0$ at each point of the solid. This assumption is classical in the analyses of localization. The elastic solid which follows such a constitutive response is also called the "linear comparison solid" (Hill, 1959). Notice that all the subsequent derivations will be carried out under this assumption, hence the equations of motion considered are:

$$\operatorname{div}\left\{(1 - D_0)\mathbb{E} : \dot{\boldsymbol{\epsilon}}(\mathbf{x}) - F(\bar{y}_0)\mathbb{E} : \boldsymbol{\epsilon}_0 \int_V \Psi(\mathbf{s})\boldsymbol{\epsilon}_0(\mathbf{s}+\mathbf{x}) : \mathbb{E} : \dot{\boldsymbol{\epsilon}}(\mathbf{s}+\mathbf{x})\,\mathrm{d}\mathbf{s}\right\} = \rho \frac{\partial^2 \mathbf{v}}{\partial t^2}. \tag{10}$$

The initial state of deformation and damage have not been specified yet. Equation (10) can be cast in a format amenable to a closed-form solution only in the particular situation where $\boldsymbol{\epsilon}_0$ and D_0 are homogeneous throughout the solid of volume V assumed to be large enough so that boundary layer effects introduced by spatial averaging can be neglected. Let us now consider the propagation of a harmonic wave in the direction defined by $\mathbf{n}$:

$$\mathbf{v}(\mathbf{x}) = \mathbf{A} \exp\left[-\mathrm{i}\xi(\mathbf{n}.\mathbf{x} - ct)\right]. \tag{11}$$

where ξ is the wave number, c is the phase velocity, $\mathbf{A}$ is the amplitude of the perturbation and i is the imaginary constant such that $i^2 = -1$. The corresponding rate of deformation is:

$$\dot{\epsilon}(\mathbf{x}) = -\tfrac{1}{2}i\xi\{\mathbf{A}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{A}\}\exp[-i\xi(\mathbf{n}.\mathbf{x}-ct)], \tag{12}$$

where $\otimes$ denotes the tensorial product. This harmonic perturbation is admissible if it satisfies the rate equation of equilibrium. Substitution of eqns (11), (12) into eqn (10) yields, under the assumption that ϵ_0 and D_0 are constant throughout the solid:

$$[(1-D_0)\mathbf{n}.\mathbb{E}.\mathbf{n}-\bar{\Psi}(\xi\mathbf{n})F(\bar{y}_0)\mathbf{n}.\mathbb{E}:\epsilon_0\otimes\epsilon_0:\mathbb{E}.\mathbf{n}].\mathbf{A} = \rho c^2\mathbf{A}, \tag{13}$$

where $\bar{\Psi}(\xi\mathbf{n})$ is defined by

$$\bar{\Psi}(\xi\mathbf{n}) = \int_V \Psi(\mathbf{s})\exp(-i\xi\mathbf{n}.\mathbf{s})\,\mathrm{d}\mathbf{s}. \tag{14}$$

Since the solid is assumed to be large, $\bar{\Psi}(\xi\mathbf{n})$ reduces to the Fourier transform of the weighting function. One should notice here that in general, $\bar{\Psi}(\xi\mathbf{n})$ depends on the direction $\mathbf{n}$ unless the weighting function is assumed to be isotropic. In the following, we will consider this last case and we will omit the dependence of $\bar{\Psi}(\xi\mathbf{n})$ on $\mathbf{n}$ and write it $\bar{\Psi}(\xi)$. Equation (13) reduces to the following eigenvalue problem:

$$[\mathbf{n}.\mathbb{H}^*(\xi).\mathbf{n}-\rho c^2\mathbb{1}].\mathbf{A} = \mathbf{0}, \tag{15}$$

where $\mathbb{1}$ is the second order identity tensor and

$$\mathbf{n}.\mathbb{H}^*(\xi).\mathbf{n} = (1-D_0)\mathbf{n}.\mathbb{E}.\mathbf{n}-\bar{\Psi}(\xi)F(\bar{y}_0)\mathbf{n}.\mathbb{E}:\epsilon_0\otimes\epsilon_0:\mathbb{E}.\mathbf{n}. \tag{16}$$

Equation (13) admits nontrivial solutions if and only if:

$$\det[\mathbf{n}.\mathbb{H}^*(\xi).\mathbf{n}-\rho c^2\mathbb{1}] = 0. \tag{17}$$

3. STATICS—SOLUTIONS AT THE BIFURCATION POINT

Let us focus now on the search for bifurcation points in statics. Besides the trivial solution in which the perturbation about the initial state ϵ_0 remains homogeneous in space, the above analysis leads to another solution to the rate equation of equilibrium when:

$$\det[\mathbf{n}.\mathbb{H}^*(\xi).\mathbf{n}] = \mathbf{0}. \tag{18}$$

This condition is similar to the necessary and sufficient condition for localization in a local continuum derived originally by Rudnicki and Rice (1975). The matrix $\mathbf{n}\mathbb{H}^*\mathbf{n}$ is similar to the acoustic tensor in the case of a local continuum. The fundamental difference is that $\mathbf{n}\mathbb{H}^*\mathbf{n}$ is dependent on the wave number of the bifurcation mode since it contains the Fourier transform of the weighting function. Equation (18) yields, after some algebra:

$$\mathbf{n}.\mathbb{E}:\epsilon_0.(\mathbf{n}.\mathbb{E}.\mathbf{n})^{-1}.\epsilon_0:\mathbb{E}.\mathbf{n} = \frac{1-D_0}{\bar{\Psi}(\xi)F(\bar{y}_0)}. \tag{19}$$

This relation is not satisfied independently from ξ, as in the local continuum. It is easy to see that the case of the local continuum is recovered in eqn (19) when $\bar{\Psi}(\xi) = 1$, that is to say when the weighting function is the Dirac distribution; in this case, the wavelength of the bifurcation mode is arbitrary. While all wavelengths are available for the local continuum in every admissible direction $\mathbf{n}$, only one wavelength is associated with each admissible direction $\mathbf{n}$ for the nonlocal continuum.

Let us particularize now the analysis to the case where the weighting function is a normalized bell-shaped function:

$$\Psi(\mathrm{x}) = \Psi_0 \exp\left(-\frac{\|x\|^2}{2l_c^2}\right), \tag{20}$$

where ψ_0 is the normalizing factor and l_c is the internal length scale of the continuum also called the characteristic length. The Fourier transform of this weighting function is:

$$\Psi(\xi\mathbf{n}) = \exp\left(-\frac{\xi^2 l_c^2}{2}\right). \tag{21}$$

Denote as $\mathbf{n}\mathbb{H}\mathbf{n}$ the acoustic tensor of the underlying local continuum, i.e. the limit of $(\mathbf{n}\mathbb{H}^*(\xi)\mathbf{n})$ when $l_c \to 0$. For every ξ, we have $\Psi(\xi) \leqslant 1$ and:

$$\det[\mathbf{n}\,.\,\mathbb{H}\,.\,\mathbf{n}] \leqslant \det[\mathbf{n}\,.\,\mathbb{H}^*(\xi)\,.\,\mathbf{n}] \tag{22}$$

with equality for ξ (or l_c) $= 0$, i.e. when the distribution of strain remains homogeneous in the nonlocal continuum which is the trivial solution. It follows that the criterion for localization (understood here as a bifurcation of the type considered above) in the local continuum is a lower bound of the criterion of localization in the corresponding nonlocal continuum. The directions $\mathbf{n}$ are also the same at the onset of bifurcation for both continua ($\xi = 0$ for the nonlocal continuum). Notice that in an independent derivation starting from a nonlocal Gurson model, Leblond *et al.* (1992) came up with the same conclusion.

Solving eqn (18) consists of finding the normal $\mathbf{n}$ and the wavelength $2\pi/\xi$ satisfying this equation. A geometrical method is used here. This method was proposed by Benallal (1992) for the analysis of localization phenomena in thermo-elasto-plasticity. Let us restrict here the investigation to the case of isotropic materials and denote by λ and μ the Lamé constants of the undamaged materials, so that

$$E_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{23}$$

One then gets

$$(\mathbf{n}\,.\,\mathbb{E}\,.\,\mathbf{n})^{-1} = \frac{1}{\mu} - \frac{\lambda+\mu}{\mu(\lambda+2\mu)}\,\mathbf{n}\otimes\mathbf{n}, \tag{24}$$

$$\mathbb{E}:\epsilon_0\,.\,\mathbf{n} = \lambda\,\mathrm{tr}\,(\epsilon_0)\mathbf{n} + 2\mu\epsilon_0\,.\,\mathbf{n}, \tag{25a}$$

$$\mathbf{n}\,.\,\mathbb{E}:\epsilon_0\,.\,\mathbf{n} = \lambda\,\mathrm{tr}\,(\epsilon_0) + 2\mu\mathbf{n}\,.\,\epsilon_0\,.\,\mathbf{n}. \tag{25b}$$

Substitution of eqns (24) and (25) into eqn (19) yields, after some algebra:

$$4\mu\mathscr{E}^2 + \frac{4\mu^2}{\lambda+2\mu}\left[e + \frac{\lambda}{2\mu}\,\mathrm{tr}\,(\epsilon_0)\right]^2 = \frac{1-D_0}{\Psi(\xi)F(\bar{y}_0)}, \tag{26}$$

where we have set:

$$e = \mathbf{n}\,.\,\epsilon_0\,.\,\mathbf{n}, \tag{27a}$$

$$\mathscr{E}^2 = (\epsilon_0\,.\,\mathbf{n})\,.\,(\epsilon_0\,.\,\mathbf{n}) - (\mathbf{n}\,.\,\epsilon_0\,.\,\mathbf{n})^2. \tag{27b}$$

$\mathscr{E}$ and e are then respectively the tangent and normal components of the strain vector in

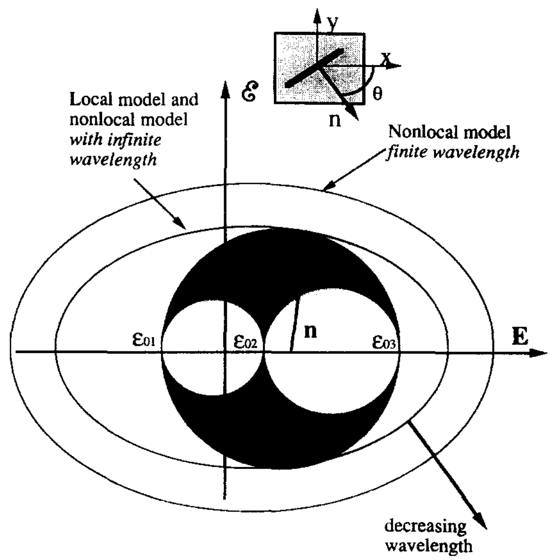

Fig. 1. Geometrical solution to the bifurcation problem: Critical state.

the direction **n**. Therefore, the initial state of deformation ϵ_0 maps the $(e, \mathscr{E})$ plane into the Mohr circles of deformation for all the directions **n** and the bifurcation criterion maps into a set of ellipses in this plane as illustrated in Fig. 1. The size of the ellipses increases as the wave number ξ decreases. Before localization, the smallest possible ellipse contains the largest Mohr circle of deformation. Bifurcation occurs first when the ellipse corresponding to $\Psi(\xi) = 1$ (smallest ellipse) is tangent to the largest Mohr circle. Obviously, this situation also provides the normal vector **n** to the localization band (Fig. 1). Figure 2 shows in the $(e, \mathscr{E})$ plane these Mohr circles and the ellipses defined by the bifurcation criterion for an initial state located beyond the critical state (i.e. first occurrence of bifurcation). When an ellipse intersects with the largest Mohr circle, there is a set of normal vectors **n** such that bifurcation is possible. However, each vector **n** corresponds to an ellipse of radius defined by $\Psi(\xi)$. In fact, the wavelength ℓ corresponding to each vector **n** is unique and it is obtained after eqn (26):

$$\ell(\mathbf{n}) = \frac{2\pi}{\xi} = \frac{\pi l_c \sqrt{2}}{\sqrt{\log\left\{\left\{4\mu\mathscr{E}^2 + \frac{4\mu^2}{\lambda+2\mu}\left[e + \frac{\lambda}{2\mu}\,\mathrm{tr}\,(\epsilon_0)\right]^2\right\}\cdot\frac{F(\tilde{y}_0)}{1-D_0}\right\}}}. \tag{28}$$

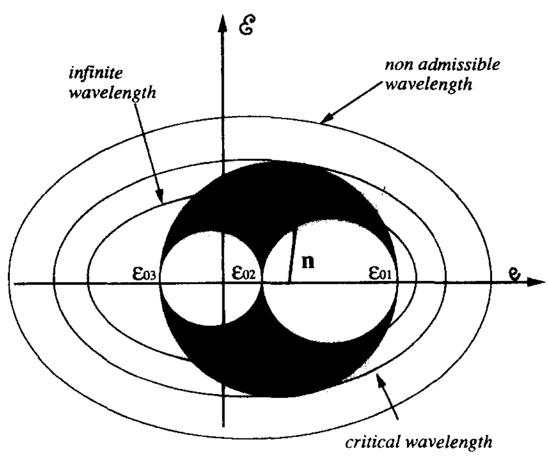

Fig. 2. Geometrical solution to the bifurcation problem: Solutions after the first bifurcation.

Thus, the role of the characteristic length becomes obvious. The wavelength of the harmonic solution is proportional to the characteristic length. This wavelength should be related to the width of the localization zone (damage band or shear band). The exact width cannot be derived from the present analysis since it results from the entire evolution process up to complete failure. Nevertheless, l_c should control the width of the localization band which grows with l_c. This observation agrees with numerical results by Bazant and Pijaudier-Cabot (1988).

In order to illustrate the present analysis, consider the initial state of deformation given by:

$$\boldsymbol{\epsilon}_0 = \begin{bmatrix} \varepsilon_{01} & 0 & 0 \\ 0 & \varepsilon_{02} & 0 \\ 0 & 0 & \varepsilon_{03} \end{bmatrix}, \tag{29}$$

with $\varepsilon_{01} \geqslant \varepsilon_{02} \geqslant \varepsilon_{03}$. The first occurrence of bifurcation is obtained when eqn (26) is satisfied, with $\Psi(\xi) = 1$. Writing that the corresponding ellipse is tangent to the largest Mohr circle leads to the localization condition for general strain and damage states:

$$4\mu\left[\left(\frac{\varepsilon_{01}-\varepsilon_{03}}{2}\right)\sin 2\theta\right]^2 + \frac{4\mu^2}{\lambda+\mu}\left[\left(\frac{\varepsilon_{01}+\varepsilon_{03}}{2}\right)\left(\frac{\varepsilon_{01}-\varepsilon_{03}}{2}\right)\cos 2\theta + \frac{\lambda}{2\mu}\operatorname{tr}(\boldsymbol{\epsilon}_0)\right]^2 - \frac{1-D_0}{F(\bar{y}_0)} = 0. \tag{30}$$

This reduces in plane strain to the localization criterion given by Pijaudier-Cabot and Bodé (1992) where it was also pointed out that under these conditions, bifurcation was excluded for situations where the nonzero principal strains have the same sign [see also Storen and Rice (1975)]. Defining the angle θ such that $\mathbf{n}^t = (\cos\theta, 0, \sin\theta)$, the normal $\mathbf{n}$ at the onset of localization can also be obtained by elementary Mohr analysis:

$$(tg2\theta)^2 = \frac{(\lambda+2\mu)F(\bar{y}_0)[\varepsilon_{01}-\varepsilon_{03}]^2-(1-D_0)}{(1-D_0)-\mu F(\bar{y}_0)[\varepsilon_{01}-\varepsilon_{03}]^2}. \tag{31}$$

It is also of interest to obtain the smallest possible wavelength $2\pi/\xi_c$ for the nonlocal model, i.e. for all possible strain and damage states. For a given state of damage and strain, the smallest available wavelength corresponds to the ellipse which is tangent to the largest Mohr circle since smaller values of $2\pi/\xi$ yields larger ellipses, and thus do not intersect with the admissible region in the Mohr plane. Using formula (30), this wavelength is then given by:

$$\Psi(\xi_c) = \frac{\dfrac{1-D_0}{F(\bar{y}_0)}}{4\mu\left[\dfrac{\varepsilon_{01}-\varepsilon_{03}}{2}\right]^2 + \dfrac{4\mu^2}{\lambda+\mu}\left[\dfrac{\varepsilon_{01}+\varepsilon_{03}}{2} + \dfrac{\lambda}{2\mu}\operatorname{tr}(\boldsymbol{\epsilon}_0)\right]^2} \tag{32}$$

and corresponds to the minimum on the right-hand side. As an illustration, let us particularize the function F as:

$$F(\bar{y}) = \frac{b_1+2b_2(\bar{y}-y^0)}{[1+b_1(\bar{y}-y^0)+b_2(\bar{y}-y^0)^2]^2}. \tag{33}$$

This function will also serve for the numerical results where the numerical values of the model parameters are E (Young modulus) = 32,000 MPa, ν (Poisson ratio) = 0.2, b_1 = 605

MPa^{-1}, $b_2 = 5.42\ 10^4\ \text{MPa}^{-2}$ and $y^0 = 60\ 10^{-6}$ MPa. The damage-dependent term in the criterion for bifurcation is:

$$\frac{1-D_0}{F(\bar{y}_0)} = \frac{1+b_1(\bar{y}-y^0)+b_2(\bar{y}-y^0)^2}{b_1+2b_2(\bar{y}-y^0)}. \tag{34}$$

As we have an infinite body, $\bar{y}_0 = y_0 = \frac{1}{2}\lambda[\text{tr}(\epsilon_0)]^2+\mu[(\epsilon_{01})^2+(\epsilon_{02})^2+(\epsilon_{03})^2]$ and it follows then that the right-hand side of (32) is a function of the three principal strains. Because of the involved symmetries, it is shown that this minimum is attained for $\varepsilon_{01} = \varepsilon_{03}$ which in turn implies that $\varepsilon_{01} = \varepsilon_{02} = \varepsilon_{03}$. Seeking the minimum for these types of loadings yields:

$$\Psi(\xi_c) = \exp\left(-\frac{\xi_c^2 l_c^2}{2}\right) = \frac{3(\lambda+\mu)}{4(3\lambda+2\mu)} \tag{35}$$

from which it follows that the smallest available bifurcation mode wavelength (for all states of strains and damage) for the nonlocal continuum is:

$$\ell_c = \frac{2\pi}{\xi_c} = \frac{2\pi l_c}{\sqrt{2\log\left[\frac{4(3\lambda+2\mu)}{3(\lambda+\mu)}\right]}}. \tag{36}$$

Studying the variations with respect to the elastic constants, one finds that this critical wavelength satisfies the conditions:

$$\frac{\pi l_c}{\sqrt{\log 2}} \leqslant \ell_c \leqslant \frac{\pi l_c}{\sqrt{\log \frac{4}{3}}}, \tag{37}$$

where the wavelength appearing in the left-hand side of the inequality corresponds to a uniaxial loading. We again see the role of the characteristic length which prevents the minimum wavelength of the localization modes from being zero. Figure 3 shows the admissible wavelengths normalized to the characteristic length l_c as a function of the initial state of damage D_0 for a uniaxial state of strain $\varepsilon_{02} = \varepsilon_{03} = -\nu\varepsilon_{01}$. At the first occurrence of bifurcation the wavelength is infinite and $\theta \approx 30°$. This angle corresponds to a direction of localization in the (x_1, x_2) or (x_1, x_3) planes. Due to the cylindrical symmetry of the problem, this angle defines a conical surface with axis x_1 on which bifurcation occurs.

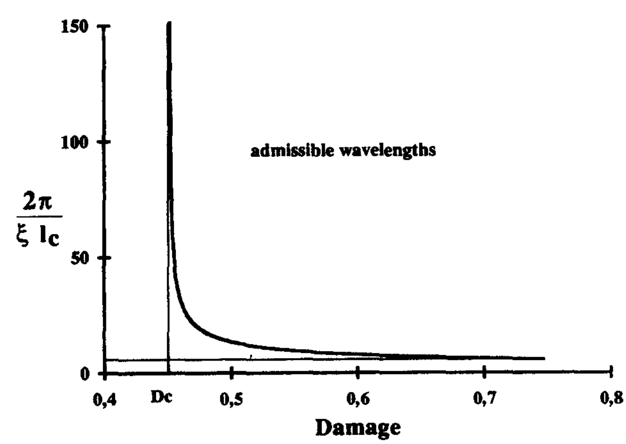

Fig. 3. Uniaxial tension: Admissible wavelengths of the localization modes as a function of the initial state of damage.

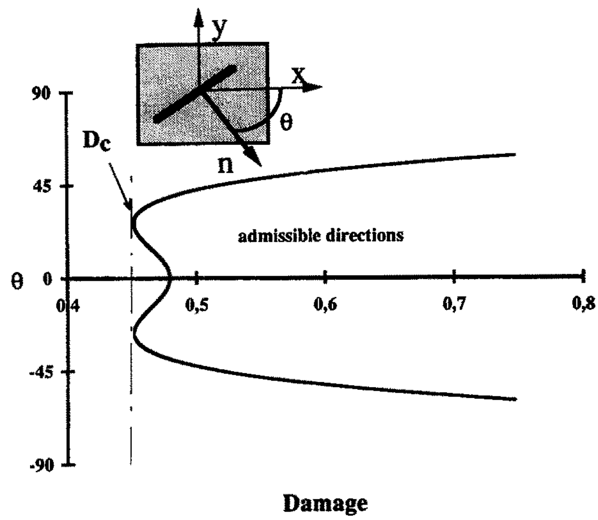

Fig. 4. Uniaxial tension: Admissible directions of the normal to the localization band as a function of the initial state of damage.

Beyond the first bifurcation, all the wavelengths higher than a critical one plotted in Fig. 3 are admissible.

Figure 4 shows the admissible directions θ of localization for the same example. In this diagram, each couple of direction $\theta \pm 45°$ corresponds to one wavelength in the plot in Fig. 3.

4. DYNAMICS—DISPERSION OF WAVES

We now turn to the dynamic problem of wave propagation controlled by the equations of motion [eqn (15)]. The classical situation for a strain-softening continuum in the one-dimensional case, is that "loading" waves can no longer propagate in the softening regime. By loading waves we mean the waves corresponding to a perturbation $\dot{\epsilon}$ that is positive (in tension) and more generally waves for which the consistency condition $\dot{f}(\bar{y}) = 0$ and $f(\bar{y}) = 0$ are simultaneously satisfied. Stationary waves appear when the hyperbolicity of the equations of motion is lost. Numerical finite elements results with the nonlocal damage model [e.g. Pijaudier-Cabot and Bazant (1987)] indicate that in the softening regime, some waves still propagate and that the width of the localization zone results from the propagation of a damage front. These characteristics will be recovered in the foregoing analytical study.

Going back to the equations of motions [eqn (15)] propagation is possible if there exists a phase velocity c, corresponding to an eigenvector $\mathbf{A}$, which is real. Thus we need to find the vanishing eigenvalues and the corresponding eigenvectors of the matrix $\mathbf{n}\,.\,\mathbb{H}^*(\xi)\,.\,\mathbf{n} - \rho c^2 \mathbb{1}$. This result can be readily obtained once the eigenvalues k_i of matrix $\mathbf{n}\mathbb{H}^*\mathbf{n}$ are computed since the phase velocities are such that $k_i = \rho c^2$. The eigenvectors $\mathbf{v}$ of the operator $\mathbf{n}\mathbb{H}^*\mathbf{n}$ are called the polarization directions. They are sought in the form:

$$\mathbf{v} = \alpha\mathbf{n} + \beta(\mathbb{E} : \epsilon_0)\,.\,\mathbf{n} + \gamma\mathbf{w}, \tag{38}$$

where $\mathbf{w}$ is orthogonal to $\mathbf{n}$ and $\mathbb{E} : \epsilon_0$. This is of course valid only when $\mathbf{n}$ and $(\mathbb{E} : \epsilon_0\,.\,\mathbf{n})$ are not colinear; we will come back to this particular case later. Constants α, β, γ are three unknown real quantities which are to be determined by:

$$[\mathbf{n}\,.\,\mathbb{H}^*(\xi)\,.\,\mathbf{n}]\,.\,\mathbf{v} = k\mathbf{v}. \tag{39}$$

Recalling that $\mathbf{n}\mathbb{H}^*\mathbf{n}$ can be written as:

$$\mathbf{n}\,.\,\mathbb{H}^*\,.\,\mathbf{n} = (1-D_0)\mu\mathbb{1} + (1-D_0)(\lambda+\mu)\mathbf{n} \otimes \mathbf{n} - \bar{\Psi}(\xi)F(\bar{y}_0)\mathbf{n}\,.\,\mathbb{E} : \epsilon_0 \otimes \epsilon_0 : \mathbb{E}\,.\,\mathbf{n} \tag{40}$$

and upon substitution of eqns (38), (40) into eqn (39), the following linear system is obtained:

$$(1-D_0)\mu\alpha+(1-D_0)(\lambda+\mu)[\alpha+\beta(\mathbf{n}.\epsilon_0:\mathbb{E}).\mathbf{n}] = k\alpha, \quad (41a)$$

$$(1-D_0)\mu\beta-(1-D_0)\Psi(\xi)F(\bar{y}_0)[\alpha(\mathbf{n}.\epsilon_0:\mathbb{E}).\mathbf{n}+\beta(\mathbf{n}.\epsilon_0:\mathbb{E}).(\mathbf{n}.\epsilon_0:\mathbb{E})] = k\beta, \quad (41b)$$

$$(1-D_0)\mu\gamma = k\gamma. \quad (41c)$$

Clearly, a first solution to eqn (39) corresponds to $\alpha = \beta = 0, \gamma$ arbitrary and to the eigenvalue $k_1 = \mu(1-D)$. It is noteworthy that the corresponding wave velocity is:

$$c = \sqrt{\frac{(1-D_0)\mu}{\rho}}, \quad (42)$$

which is the velocity of tangential waves in an elastic medium with Lamé constants $(1-D)\lambda$ and $(1-D)\mu$. Furthermore, this result also holds for the underlying local continuum. The direction of polarization of the corresponding wave is orthogonal to $\mathbf{n}$ and to $(\mathbb{E}:\epsilon_0.\mathbf{n})$. The two other eigenvalues of the marix $\mathbf{n}\mathbb{H}^*\mathbf{n}$ are obtained from eqns (41a, b), which is a system in (α, β) which must be singular in order to admit a nontrivial solution. Therefore k_2, k_3 are solutions of the equation:

$$\det\begin{bmatrix} (1-D_0)(\lambda+2\mu)-k & (1-D_0)(\lambda+\mu)\mathbf{n}.(\mathbb{E}:\epsilon_0).\mathbf{n} \\ -\Psi(\xi)F(\bar{y}_0)\mathbf{n}.(\mathbb{E}:\epsilon_0.\mathbf{n}) & (1-D_0)\mu-\Psi(\xi)F(\bar{y}_0)(\mathbb{E}:\epsilon_0.\mathbf{n})^2-k \end{bmatrix} = 0, \quad (43)$$

which can be recombined in terms of $\mathscr{E}^2$ and ϵ^2 defined before. After some algebra, the determinant reduces to:

$$k^2-k[\{(M_1)^2+(M_2)^2\}\Psi(\xi)F(\bar{y}_0)-(1-D_0)(\lambda+3\mu)]$$
$$-[(M_1)^2(\lambda+2\mu)+\mu(M_2)^2](1-D_0)\Psi(\xi)F(\bar{y}_0)+(1-D_0)^2\mu(\lambda+2\mu) = 0, \quad (44)$$

where we have set

$$(M_1)^2 = 4\mu^2\mathscr{E}^2, \quad (M_2)^2 = 4\mu^2\left[\epsilon+\frac{\lambda}{2\mu}\operatorname{tr}(\epsilon_0)\right]^2. \quad (45)$$

When $\xi \to \infty$, i.e. for waves with vanishing wavelengths, eqn (42) admits the two real solutions

$$k_{1\infty} = (1-D_0)\mu, \quad k_{2\infty} = (1-D_0)(\lambda+2\mu), \quad (46)$$

which correspond again to the velocities of waves propagating in the elastic-damaged medium. Hence, one expects that sufficiently short wavelength modes always propagate in the medium. It is important to know in which situation the phase velocity of a given wave may become imaginary. For this, consider the discriminant of eqn (44):

$$\Delta = [\{(M_1)^2+(M_2)^2\}\Psi(\xi)F(\bar{y}_0)]^2-(1-D_0)(\lambda+3\mu)]$$
$$+2(1-D_0)(\lambda+\mu)[(M_1)^2-(M_2)^2]\Psi(\xi)F(\bar{y}_0)+(1-D_0)^2(\lambda+\mu)^2. \quad (47)$$

At a given state ϵ_0 and for a given direction $\mathbf{n}$, i.e. for fixed values of M_1 and M_2, Δ can be viewed as a second degree polynomial in terms of $\Psi(\xi)F(\bar{y}_0)$. In order to decide whether Δ is positive (real or pure imaginary velocities) or negative (complex velocities), we look at whether it is possible to find $\Psi(\xi)F(\bar{y}_0)$ such that Δ vanishes. The discriminant of the equation $\Delta = 0$ is:

$$\Delta' = -16(1-D)^2(\lambda+\mu)^2(M_1)^2(M_2)^2 \tag{48}$$

and it is always negative. Thus, Δ is either positive or zero when M_1 (or M_2) vanishes. Note that when $M_1 = 0$, $\mathbf{n}$ and $\mathbb{E}:\boldsymbol{\epsilon}_0.\mathbf{n}$ are colinear and the present analysis does not hold. Therefore, the eigenvalues k_2 and k_3 are always real. Next, eqn (44) shows that:

$$k_2k_3 = (1-D_0)^2\mu(\lambda+2\mu)-[(M_1)^2(\lambda+2\mu)+\mu(M_2)^2](1-D_0)\Psi(\xi)F(\bar{y}_0). \tag{49}$$

When this product vanishes, we have $c = 0$ (i.e. stationary waves) and this corresponds to the localization condition in statics analysed in the previous section. The sign of this product k_2k_3 can then be obtained by the geometrical interpretation of the static localization criterion given before in the $(e, \mathscr{E})$ plane. If the point corresponding to $\mathbf{n}$ in the Mohr plane lies outside the ellipse defining the static localization criterion for a given ξ, then $k_3k_2 < 0$. Conversely if it is located inside this ellipse, k_2 and k_3 have the same sign. In this last case, the sign of the sum k_2+k_3 can be obtained from (43):

$$k_2+k_3 = (1-D_0)(\lambda+3\mu)-\{(M_1)^2+(M_2)^2\}\Psi(\xi)F(\bar{y}_0). \tag{50}$$

When $k_2k_3 > 0$ (i.e. the roots have the same sign):

$$\Psi(\xi)F(\bar{y}_0) < \frac{(1-D_0)\mu(\lambda+2\mu)}{(M_1)^2(\lambda+2\mu)+\mu(M_2)^2} \tag{51}$$

reporting this result in (50) yields:

$$k_2+k_3 > (1-D_0)\frac{(M_1)^2(\lambda+2\mu)^2+\mu^2(M_2)^2}{(M_1)^2(\lambda+2\mu)+\mu(M_2)^2}. \tag{52}$$

Hence, when k_2 and k_3 have the same sign, the two eigenvalues are always positive because the right-hand side of (52) is positive.

To summarize, two situations may occur in general:

(1) The two roots of (44) have the same sign and eqn (39) leading to the phase velocities has three real solutions, the phase velocities are real.

(2) These two roots have opposite signs and eqn (39) has two real solutions and one which is purely imaginary, one phase velocity is imaginary.

Figure 5 summarizes these results: at ξ fixed and if the normal $\mathbf{n}$ is such that $(\boldsymbol{\epsilon}_0.\mathbf{n})$ lies inside the ellipse corresponding to bifurcation in statics, k_1, k_2, k_3 are positive and the

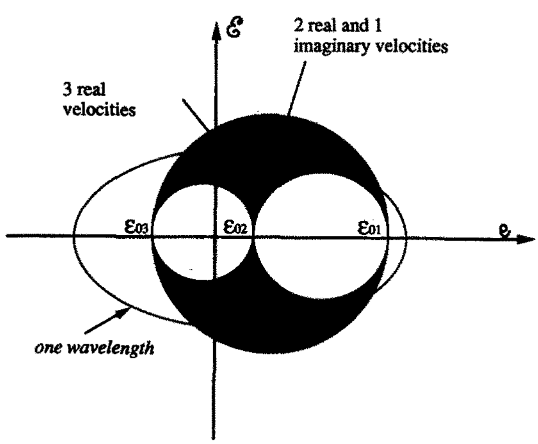

Fig. 5. Geometrical solution in dynamics: Nature of the three phase velocities as a function of the direction $\mathbf{n}$ of propagation of the wave.

corresponding phase velocities are real. Otherwise there exists one phase velocity which becomes imaginary.

Different types of waves are available: those corresponding to k_1 are classical tangential waves and are not dispersive since the phase velocity is independent of the wave number; those corresponding to k_2 and k_3 are in general dispersive as waves with different wave numbers propagate with a different velocity since the roots k_2, k_3 are functions of ξ [see eqn (44)]. This characteristic was also recovered by Sluys *et al.* (1991) using gradient dependent plasticity and by De Borst *et al.* (1992) for a Cosserat continuum, who pointed out the importance of wave dispersion in localization analyses.

Given one direction of propagation, one of the three wave components may not propagate in the nonlocal continuum. The critical wave number ξ_c, above which any wave component can propagate in any aribtrary direction, is derived from the condition $k_2 k_3 = 0$ [which is the static localization criterion (26)]. This provides a critical wavelength equal to that in eqn (28) (or a cutting frequency) above which some waves may possess an imaginary velocity. In any instance, short-wavelength modes always propagate in the softening regime. These waves yield to a moving damage front, the displacement of which can be used as an indication for mesh refinement in transient finite element studies (Huerta *et al.*, 1992).

To be complete, we come back now to the case where $\mathbf{n}$ and $\mathbb{E}:\boldsymbol{\epsilon}_0.\mathbf{n}$ are colinear as the above analysis does not hold. Let $\mathbb{E}:\boldsymbol{\epsilon}_0.\mathbf{n} = \omega\mathbf{n}$, which means that $\lambda\,\mathrm{tr}(\boldsymbol{\epsilon}_0)\mathbf{n} + 2\mu\boldsymbol{\epsilon}_0.\mathbf{n} = \omega\mathbf{n}$ and consequently:

$$\boldsymbol{\epsilon}_0 \cdot \mathbf{n} = \frac{[\omega - \lambda\,\mathrm{tr}(\boldsymbol{\epsilon}_0)]}{2\mu} \cdot \mathbf{n} \tag{53}$$

and thus $\mathbf{n}$ is a principal direction of the strain tensor. In this case:

$$\mathbf{n}.\mathbb{H}^*.\mathbf{n} = (1-D_0)\mu\mathbb{1} + \{(1-D_0)(\lambda+\mu) - \omega^2\Psi(\xi)F(\bar{y}_0)\}\mathbf{n}\otimes\mathbf{n}. \tag{54}$$

$k_1 = (1-D_0)\mu$ is an eigenvalue of order two (with corresponding eigenvectors all vectors of the plane orthogonal to $\mathbf{n}$). The third eigenvalue is obtained by taking the trace of the matrix in eqn (40) and subtracting $2(1-D_0)\mu$.

A simple example of this situation is the case of a uniaxial test and a wave propagating in the direction of the loading:

$$\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{with} \quad \boldsymbol{\epsilon}_0 = \begin{bmatrix} \varepsilon_{01} & 0 & 0 \\ 0 & -\nu\varepsilon_{01} & 0 \\ 0 & 0 & -\nu\varepsilon_{01} \end{bmatrix}.$$

The eigenvalues of $(\mathbf{n}\mathbb{H}^*\mathbf{n})$ are:

$$k_1 = k_2 = (1-D_0)\mu, \tag{55a}$$

$$k_3 = (1-D_0)(\lambda+2\mu) - \omega^2\Psi(\xi)F(\bar{y}_0). \tag{55b}$$

Only the longitudinal wave is dispersive. The critical wave length for which $c_3 = \sqrt{k_3/\rho} = 0$ is a solution of:

$$\frac{(1-D_0)(\lambda+2\mu)}{E\varepsilon_{01}\bar{y}_0} = \Psi(\xi). \tag{56}$$

Similar results were obtained in a one-dimensional analysis (Pijaudier-Cabot *et al.*, 1992). The transverse waves (corresponding to k_1) are not polarized and only the longi-

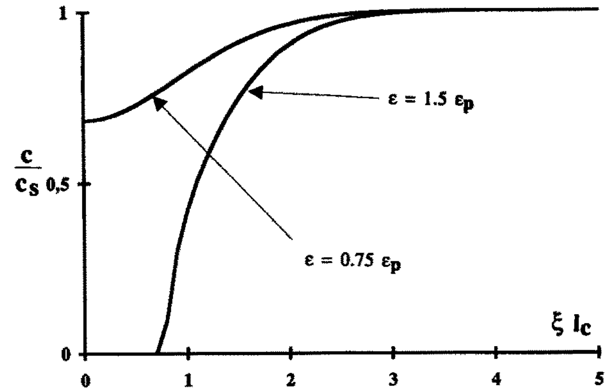

Fig. 6. Uniaxial tension in direction x_1: Velocity of the longitudinal wave propagating in direction x_1 as a function of the wavelength.

tudinal wave (corresponding to k_3) is dispersive. Figure 6 shows the evolution of the wave speeds for different wave numbers, before the initial state of static deformation is such that singularity of the pseudo-acoustic tensor in eqn (16) is reached and after. We can see that the phase velocity is bounded by the velocity of the elastic (unloading) waves:

$$c_s = \sqrt{\frac{(\lambda+2\mu)(1-D_0)}{\rho}}.$$

At this point, it should be suggested that such a formula could be applied in an experimental analysis for obtaining the values of $\bar{\Psi}(\xi)$ and subsequently the internal length of the material and the weight function Ψ if different states of initial strain ε_0 are considered. The existence of a cut off frequency in the spectrum of propagating waves should also be related to the internal structure of the material and to the size of the heterogeneities (spherical voids in this case since damage is isotropic). In fact it ought to be connected to wave scattering (Piau, 1980) due to the heterogeneities. Let us finally stress that dispersion of elastic waves was among the most important arguments for introducing nonlocal elasticity (Eringen and Edelen, 1972) instead of classical elasticity. Experiments on initially damaged material could very well provide information as far as the usefulness of nonlocal models with local strain is concerned.

Another particular case corresponds to a wave propagating in a direction which is orthogonal to the axis 1 ($M_1 = 0$):

$$\mathbf{n} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The eigenvalues of $\mathbf{n}\mathbb{H}^*\mathbf{n}$ are again trivial

$$k_1 = k_2 = (1-D)\mu$$
$$k_3 = (1-D)(\lambda+2\mu). \tag{57}$$

Waves propagating in this direction are not dispersive, their speed is that of elastic waves.

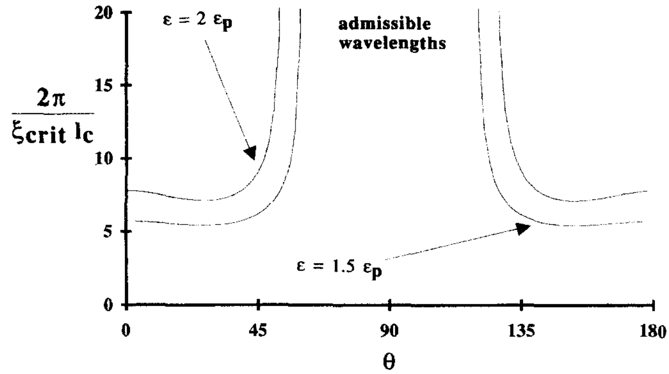

Fig. 7. Uniaxial tension in direction x_1: Critical wavelength of the first stationary waves propagating in the plane (x_1, x_2), $x_3 = 0$.

Figure 7 shows the evolution of the critical wavelength as a function of the angle θ defining the normal **n** in the (x_1, x_2) plane with $x_3 = 0$. The calculation was made with two values of the initial state of deformation (uniaxial state of stress $\sigma_{01} \neq 0$) $\varepsilon_{01} = 1.5\ \varepsilon_p$ and $\varepsilon_{01} = 2\ \varepsilon_p$ where ε_p is the strain at peak stress. As expected, there is a set of directions for which all the waves have a real phase velocity. In the present example, some waves may become stationary only when their direction of propagation is close enough to the load direction.

5. CONCLUSIONS

The results obtained from this analysis highlight the role of the weighting function and of the internal length. Several conclusions can be drawn.

(1) The criterion for the occurrence of bifurcation obtained for the nonlocal model is very similar to that of the local continuum following the same damage evolution law. This is obtained under the condition that the initial state around which the perturbation analysis is performed is homogeneous. The necessary and sufficient condition for localization in the local continuum is a lower bound of the criterion for bifurcation in the nonlocal continuum.

(2) The local continuum acts as a lower bound of the nonlocal continuum as far as localization, understood as a bifurcation into harmonic modes of deformation, is concerned. Furthermore, the orientation of the localization band is the same. At the onset of bifurcation, the boundary value problem for the *local* continuum becomes ill posed since the wavelength of the localized modes is arbitrary. In the *nonlocal* continuum, only one wavelength is allowed. This is due to the presence of the Fourier transform of the weight function in the expression of the operator [**n**ℍ***n**]. Detailed studies on the well-posedness of the boundary value problem for finite bodies with the nonlocal model are in progress.

(3) For each possible orientation of the localization band in the nonlocal continuum, there exists one suitable wavelength which is proportional to the characteristic length of the material. This wavelength cannot be below a certain threshold which justifes totally the appellation "Localization Limiter" used for this type of constitutive relation.

(4) Wave propagation in this nonlocal continuum is dispersive. One of the three possible phase velocities, derived from the equations of motion, becomes imaginary in the softening regime. Nevertheless, waves with a sufficiently short wave length can always propagate. Wave dispersion ought to be combined in future studies with the micro-structure of the continuum and in particular with scattering of elastic waves in a material containing voids. These results may provide valuable information on the possible relationship between the characteristic length of the nonlocal continuum and the size of the voids.

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